

S_3 symmetry and the quark mixing matrix

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Abstract

We impose an S_3 symmetry on the quark fields under which two of three quarks transform like a doublet and the remaining one as singlet, and use a scalar sector with the same structure of $SU(2)$ doublets. After gauge symmetry breaking, a \mathbb{Z}_2 subgroup of the S_3 remains unbroken. We show that this unbroken subgroup can explain the approximate block structure of the CKM matrix. By allowing soft breaking of the S_3 symmetry in the scalar sector, we show that one can generate the small elements, of quadratic or higher order in the Wolfenstein parametrization of the CKM matrix. We also predict the existence of exotic new scalars, with unconventional decay properties, which can be used to test our model experimentally.

Because of the discovery of a boson of mass about 126 GeV at the Large Hadron Collider (LHC) [1,2], we are convinced of existence of elementary bosons of spin not equal to 1. The discovery opens up the question whether there are more particles of the same kind. It is of course an experimental question, but the possibility can be made attractive from a theoretical standpoint if one can show that additional particles can help us understand some properties or relate different parameters of the Standard Model (SM). This paper is an attempt in that direction. We show that the structure of the Cabibbo-Kobayashi-Maskawa (CKM) matrix can be understood through an extended scalar sector and the breaking of a discrete symmetry.

A very useful parametrization of the CKM matrix was given by Wolfenstein [3], which shows a hierarchical pattern of the elements of the matrix. We will show that this pattern can be related, through an extended Higgs sector, to the breaking of a discrete S_3 symmetry that we impose on the quartic terms of the Lagrangian. Admittedly, there have been many attempts to explain the quark sector using S_3 symmetry [4–11]. But in most of them [5,6,8,11] convenient relations among the VEVs of different scalar multiplets have been assumed along with the assumption that the scalar potential can produce such relations. On the contrary, in a previous paper [12] we studied in detail the *most general* S_3 symmetric scalar potential with three Higgs doublets and found that a \mathbb{Z}_2 subgroup of the S_3 remains intact after the spontaneous symmetry breaking. In this paper, we use this remnant \mathbb{Z}_2 symmetry to explain the mass and mixing patterns in the quark sector.

The discrete symmetry group S_3 has two 1-dimensional and one 2-dimensional irreducible representations, which we will denote by $\mathbf{1}$, $\mathbf{1}'$ and $\mathbf{2}$. We pick a basis such that the generators of the S_3 group in the $\mathbf{2}$ representation is given by,

$$a = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}, \quad b = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}. \quad (1)$$

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Note that a is of order 3, whereas b is of order 2. The rest of the elements can be obtained by taking products of powers of these two elements. In this basis the quark fields are assigned the following representations of S_3 :

$$\mathbf{2} : \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}, \begin{bmatrix} u_{1R} \\ u_{2R} \end{bmatrix}, \begin{bmatrix} d_{1R} \\ d_{2R} \end{bmatrix}, \quad (2a)$$

$$\mathbf{1} : Q_3, u_{3R}, d_{3R}. \quad (2b)$$

where the Q_i 's are the usual left-handed $SU(2)$ quark doublets, whereas the u_{iR} 's and d_{iR} 's are the right-handed up-type and down-type quark fields which are singlets of the $SU(2)$ part of the gauge symmetry. Note that the square brackets, as in Eq. (1) as well, denote the doublet representation of S_3 , and has nothing to do with the representation of the enclosed fields under $SU(2)$. Similarly, in the Higgs sector, there are three $SU(2)$ doublets ϕ_i ($i = 1, 2, 3$), and their transformation under the S_3 symmetry is as follows:

$$\mathbf{2} : \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \equiv \Phi, \quad \mathbf{1} : \phi_3. \quad (3)$$

The most general scalar potential obeying gauge symmetry and S_3 symmetry has been given by many authors [8, 12–18], and there is no need to repeat the expression here. If we assume that all scalar couplings allowed by the aforesaid symmetries are non-zero, that all VEVs are real in order to avoid CP violation in the scalar potential, then the minimization of the potential yields the relation [12]

$$v_1 = \sqrt{3}v_2 \quad (4)$$

assuming that the values of the parameters in the potential are such that this minimum is favored over another one which has $v_3 = 0$. Of course, $v = \sqrt{v_1^2 + v_2^2 + v_3^2} = 246$ GeV appears in the masses of the W and Z bosons. With the VEV relation in Eq. (4), one can obtain an *alignment limit* [12] where one of the CP-even Higgs bosons, h , will have SM-like tree-level couplings with the SM particles and therefore the LHC Higgs data can be explained. The important consequence of Eq. (4) is that a \mathbb{Z}_2 symmetry survives the spontaneous symmetry breaking, a symmetry that is generated by the element which inflicts the transformation

$$\Phi \rightarrow b\Phi \quad (5)$$

with b defined in Eq. (1), since this transformation leaves the VEVs unaffected. Note that the VEV relation of Eq. (4) depends on our choice of the basis for the doublet representation of S_3 . In another equivalent doublet representation of S_3 , the relation between the VEVs will change and the elements of a and b in Eq. (1) will also change accordingly, but the vacuum will still remain invariant under the transformation of Eq. (5). In other words, the existence of a remnant \mathbb{Z}_2 symmetry does not depend on the choice of basis, and it is this fact that contains the essential physics, as we will see shortly.¹

We now present the most general Yukawa couplings involving the u_R quarks that is consistent with the gauge and S_3 symmetries.

$$\begin{aligned} \mathcal{L}_Y^{(u)} = & -y_1^u (\bar{Q}_1 \tilde{\phi}_3 u_{1R} + \bar{Q}_2 \tilde{\phi}_3 u_{2R}) - y_2^u \left\{ (\bar{Q}_1 \tilde{\phi}_2 + \bar{Q}_2 \tilde{\phi}_1) u_{1R} + (\bar{Q}_1 \tilde{\phi}_1 - \bar{Q}_2 \tilde{\phi}_2) u_{2R} \right\} \\ & - y_3^u \bar{Q}_3 \tilde{\phi}_3 u_{3R} - y_4^u \bar{Q}_3 (\tilde{\phi}_1 u_{1R} + \tilde{\phi}_2 u_{2R}) - y_5^u (\bar{Q}_1 \tilde{\phi}_1 + \bar{Q}_2 \tilde{\phi}_2) u_{3R} + \text{h.c.} \end{aligned} \quad (6)$$

We have used the convention in which the lower component of the $SU(2)$ doublets of Higgs multiplets are uncharged, and used the standard abbreviation $\tilde{\phi}_i = i\sigma_2 \phi_i^*$. The Yukawa couplings of the d_R quarks can be obtained by replacing u_{iR} by d_{iR} , y_i^u by y_i^d , and $\tilde{\phi}_i$ by ϕ_i in Eq. (6). The Yukawa couplings are in general complex, which can be responsible for CP violation. It should be noted that the fields u_i and d_i presented here do not represent physical quark fields. Their superpositions which are eigenstates will be given later.

After symmetry breaking, the mass matrix that arises in the up-type quark sector is the following:

$$\mathcal{M}_u = \begin{pmatrix} y_1^u v_3 + y_2^u v_2 & y_2^u v_1 & y_5^u v_1 \\ y_2^u v_1 & y_1^u v_3 - y_2^u v_2 & y_5^u v_2 \\ y_4^u v_1 & y_4^u v_2 & y_3^u v_3 \end{pmatrix}. \quad (7)$$

¹We note that the choice $v_3 = 0$ with $v_2 = \sqrt{3}v_1$ can lead to interesting dark matter candidates.

This matrix can be easily block-diagonalized. Taking

$$X = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (8)$$

we find that

$$\mathcal{M}_u^{\text{block}} \equiv X \mathcal{M}_u X^\dagger = \begin{pmatrix} y_1^u v_3 - 2y_2^u v_2 & 0 & 0 \\ 0 & y_1^u v_3 + 2y_2^u v_2 & 2y_5^u v_2 \\ 0 & 2y_4^u v_2 & y_3^u v_3 \end{pmatrix}, \quad (9)$$

The mass matrix for the down-type quark is obtained by replacing all y^u 's by the corresponding y^d 's. It can also be block-diagonalized using the same matrix X .

We can now identify the singleton blocks of the mass matrices to be the masses of the third generation of quarks. For example, the t -quark mass will be given by

$$m_t = |y_1^u v_3 - 2y_2^u v_2|, \quad (10)$$

and a similar equation for m_b . In other words, starting from the original basis of quark fields that we had denoted by u_1, u_2, u_3 , we have reached a new basis defined by

$$\begin{pmatrix} t \\ c' \\ u' \end{pmatrix} = X \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}. \quad (11)$$

The reason for the block-diagonal nature of the matrix in Eq. (9) can be understood very easily from this new basis. Notice that Eq. (11) implies that

$$t = \frac{1}{2}(u_1 - \sqrt{3}u_2). \quad (12)$$

It can be easily checked, using the \mathbb{Z}_2 generator, b , that appears in Eq. (1), that this combination changes sign under the remnant \mathbb{Z}_2 transformation, i.e., it is \mathbb{Z}_2 -odd. The other two members in the new basis, c' and u' , are \mathbb{Z}_2 -even. Because the \mathbb{Z}_2 symmetry remains intact, there is no mixing between states which are odd under it with states which are even.

This block structure has a very important consequence on the CKM matrix, which is the main point of our article. In order to obtain the physical eigenstates, we still need to further rotate the 2×2 block that remains in $\mathcal{M}^{\text{block}}$. In other words, we can find a bi-unitary transformation such that

$$U_L^\dagger \mathcal{M}_u^{\text{block}} U_R = \mathcal{M}_u^{\text{diag}} = \text{diag}(m_t, m_c, m_u). \quad (13)$$

Both U_L and U_R would be block-diagonal. We can take U_L to be of the form

$$U_L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_u & -\sin \theta_u \\ 0 & \sin \theta_u & \cos \theta_u \end{pmatrix}, \quad (14)$$

with the understanding that all phases can be absorbed in U_R . Therefore, combining Eqs. (9) and (13), we obtain that

$$\mathcal{M}_u^{\text{diag}} = \mathcal{U}_L^\dagger \mathcal{M}_u \mathcal{U}_R, \quad (15)$$

where

$$\mathcal{U}_L = X^\dagger U_L. \quad (16)$$

The relation between original states d_{iL} ($i = 1, 2, 3$) and the mass eigenstates in the down sector will have a similar form, governed by the matrix

$$\mathcal{D}_L = X^\dagger D_L, \quad (17)$$

where D_L is a matrix like U_L , except with a different angle θ_d . The CKM matrix is then given by²

$$V_{\text{CKM}} = \mathcal{U}_L^\dagger \mathcal{D}_L = \begin{matrix} & \mathbf{b} & \mathbf{s} & \mathbf{d} \\ \begin{matrix} \mathbf{t} \\ \mathbf{c} \\ \mathbf{u} \end{matrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_C & -\sin \theta_C \\ 0 & \sin \theta_C & \cos \theta_C \end{pmatrix} \end{matrix}, \quad (18)$$

where $\theta_C = \theta_d - \theta_u$.

There are some interesting points to note here. First, the CKM matrix does not depend on the matrix X that was used only to define an intermediate basis to understand the effect of the remnant \mathbb{Z}_2 symmetry. Second, with the \mathbb{Z}_2 symmetry intact, the CKM matrix is block diagonal. This is expected, since the W -boson, being \mathbb{Z}_2 -even [12], cannot couple a \mathbb{Z}_2 -even quark to another which is \mathbb{Z}_2 -odd. This conclusion, i.e., the zeros of the CKM matrix that appears in Eq. (18), will not be modified by loop corrections because of the unbroken \mathbb{Z}_2 symmetry.

In order to obtain the realistic CKM matrix, one therefore has to break the S_3 symmetry in the Lagrangian itself. We assume that there are soft terms in the scalar potential which are not S_3 symmetric. For example, we can consider

$$V_{\text{soft}} = \mu_{13}^2 (\phi_1^\dagger \phi_3 + \phi_3^\dagger \phi_1), \quad (19)$$

with μ_{13}^2 being much smaller than the other bilinear parameters. The presence of this term will slightly modify the VEV relation of Eq. (4). Let us denote the changed relation by

$$v_1 = \sqrt{3}v_2 + \Delta, \quad (20)$$

where $\Delta \ll v_2$. This deviation Δ from Eq. (4) is what is important, and not the details of the soft-breaking terms, since the generic form of Eq. (20) is obtained even if we make some other choices in Eq. (19). The modified mass matrix in the up sector is therefore given by

$$\widetilde{\mathcal{M}}_u = \mathcal{M}_u + \begin{pmatrix} 0 & y_2^u \Delta & y_5^u \Delta \\ y_2^u \Delta & 0 & 0 \\ y_4^u \Delta & 0 & 0 \end{pmatrix}. \quad (21)$$

We now need to modify the diagonalizing matrices. Instead of the prescription of Eq. (15), we will now need to use

$$\mathcal{M}_u^{\text{diag}} = \mathcal{U}_L^\dagger \widetilde{\mathcal{M}}_u \mathcal{U}_R, \quad (22)$$

where we can define \mathcal{U}_L (similarly for \mathcal{U}_R also) in the form

$$\mathcal{U}_L = \mathcal{U}_L \mathfrak{U}_L, \quad (23)$$

where \mathfrak{U}_L is close to the unit matrix which takes into account the effect of very small Δ in Eq. (20). The task now is to determine the form of this matrix.

For this, let us look back at Eq. (9), in particular at the first two diagonal elements of the matrix. The top quark mass is given in Eq. (10), which has to be large. One can ask which of the two terms dominates in the

²The first column and the first row of the CKM matrix correspond to the b and the t quarks merely because we do not want to disturb the notation of Ref. [12]. We could have easily taken u_2 and u_3 as part of an S_3 doublet in Eq (2), and then we could have put their \mathbb{Z}_2 -odd combination in the third row.

expression. If any one term is considerably larger in absolute value than the other term, the 22-element of the matrix $\mathcal{M}_u^{\text{block}}$ will roughly be equal to the 11-element. That would be a disaster, because the trace of the lower 2×2 block, which should be of the order of charm quark mass, would be then close to m_t in absolute value. The only way this problem can be avoided, i.e., the 22-element remains much smaller than the 11-element, is by having both terms almost equal in magnitude, so that their magnitudes add up in m_t but largely cancel in the 22-element. This means that, with comparable VEVs, both y_1^u and y_2^u will have to be larger than the other Yukawa couplings by about an order of magnitude. Thus the 12 and 21 elements of the correction matrix of Eq. (21) are much larger than the 13 and 31 elements. So we anticipate a form for the correction matrix \mathfrak{U}_L where the 13 and 31 elements will be down by a power of some small (but not very small) parameter λ , which will be specified shortly.

There will be a similar correction matrix \mathfrak{D}_L coming from the down sector. If, for the moment, we assume that \mathfrak{D}_L is equal to the unit matrix to the accuracy desired, we can write the corrected CKM matrix as

$$V_{\text{CKM}} = \mathfrak{U}_L^\dagger \mathcal{U}_L^\dagger \mathcal{D}_L. \quad (24)$$

We mentioned earlier that \mathfrak{U}_L involves some small parameter λ . Following Wolfenstein [3], we can use the Cabibbo angle for this parameter. We define

$$\lambda = \sin \theta_C, \quad (25)$$

and, motivated by the form for the correction matrix in Eq. (21), write

$$\mathfrak{U}_L = \begin{pmatrix} 1 & A\lambda^2 & C\lambda^3 \\ A'\lambda^2 & 1 & 0 \\ C'\lambda^3 & 0 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4), \quad (26)$$

which is consistent with our order of magnitude estimations in the previous paragraphs.³ We can choose the phases of the quark fields such that the co-efficient A is real in Eq. (26). The unitarity of this matrix is achieved, to the same accuracy in λ , by choosing

$$A' = -A, \quad C' = -C^*. \quad (27)$$

In order to maintain consistency, we need to use the same accuracy of λ for the \mathbb{Z}_2 invariant CKM matrix given in Eq. (18). Thus we obtain, correct up to terms of $\mathcal{O}(\lambda^3)$, the following form for the CKM matrix:

$$V_{\text{CKM}} = \begin{pmatrix} 1 & -A\lambda^2 & -C\lambda^3 \\ A\lambda^2 & 1 & 0 \\ C^*\lambda^3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - \frac{\lambda^2}{2} & -\lambda \\ 0 & \lambda & 1 - \frac{\lambda^2}{2} \end{pmatrix} + \mathcal{O}(\lambda^4) \quad (28)$$

$$= \begin{matrix} & \mathbf{b} & \mathbf{s} & \mathbf{d} \\ \begin{matrix} \mathbf{t} \\ \mathbf{c} \\ \mathbf{u} \end{matrix} & \begin{pmatrix} 1 & -A\lambda^2 & A\lambda^3(1 - \rho - i\eta) \\ A\lambda^2 & 1 - \frac{\lambda^2}{2} & -\lambda \\ A\lambda^3(\rho - i\eta) & \lambda & 1 - \frac{\lambda^2}{2} \end{pmatrix} \end{matrix} + \mathcal{O}(\lambda^4), \quad (29)$$

where in the last step, the CKM matrix in terms of the Wolfenstein parameters are obtained by choosing

$$C = (\rho + i\eta)A. \quad (30)$$

So far we have assumed that \mathfrak{D}_L is the unit matrix. This is not an unjust assumption. The reason is that the mass hierarchy in the down sector is not so violent as in the up-sector. So in the expression for $\mathcal{M}_d^{\text{block}}$ which is

³Had we replaced v_2 through Eq. (20) in the expression for $\widetilde{\mathcal{M}}_u$, the form for the correction matrix would have been different. That would have obscured much of the subsequent discussions, although the final result should have been the same because physics should be independent of the parametrization.

obtained by replacing y^u 's with y^d 's in $\mathcal{M}_u^{\text{block}}$, y_1^d can carry almost all the bottom quark mass while y_2^d is very small. Since y_4^d and y_5^d only get involved into the expressions for strange and down quark masses, they, too, are expected to be very small. Therefore, the overall size of the perturbation matrix for the down sector can be considered to be much smaller compared to that for the up sector. Hence at the leading order, \mathfrak{D}_L can be approximated as a 3×3 unit matrix. But even if we consider the small departures of \mathfrak{D}_L from the unit matrix, we can use a form like that for \mathfrak{U}_L shown in Eq. (23) with A and C replaced by different parameters. Then the CKM matrix will be given by

$$V_{\text{CKM}} = \mathfrak{U}_L^\dagger \mathcal{U}_L^\dagger \mathcal{D}_L \mathfrak{D}_L. \quad (31)$$

However, this will not change the general form of the CKM matrix shown in Eq. (29). The Cabibbo block will not change at all to the order shown and the Wolfenstein parameters, A , ρ and η will be defined by linear combinations of parameters appearing in \mathfrak{U}_L and \mathfrak{D}_L .

To conclude, we have shown that the quark masses and mixings can be understood from a Lagrangian with an S_3 symmetry, broken spontaneously down to \mathbb{Z}_2 by the VEVs that break gauge symmetry. Regarding the masses, our analysis provides an explanation of the third generation of quarks being widely different from the first two — the third generation is \mathbb{Z}_2 -odd whereas the first two are \mathbb{Z}_2 -even. Regarding mixing, we obtain the Wolfenstein form of the CKM matrix. Wolfenstein fixed the orders of λ in various elements of the CKM matrix by experimental data only. In our case, we show that the remnant \mathbb{Z}_2 symmetry ensures the form of the mixing matrix to $\mathcal{O}(\lambda)$. Our corrections to this order were motivated by consideration of a *heavy* top quark mass, and by terms which softly break the S_3 symmetry. Without these soft breaking terms, the third generation quarks do not mix at all with the other two generations. Therefore, smallness of $b \rightarrow s\gamma$ branching ratio can be related, in the 't Hooft sense of naturalness, to the smallness of the soft breaking terms.

Tests of the idea presented here would consist of checking consequences of the \mathbb{Z}_2 symmetry. In the limit that the \mathbb{Z}_2 symmetry is exact, in addition to the SM-like Higgs h which is \mathbb{Z}_2 -even, there will be four neutral and two pairs of charged spinless particles. Among these, the scalar h^0 , the pseudoscalar A_1 and one pair of charged scalars H_1^\pm would be \mathbb{Z}_2 -odd, and the others will be \mathbb{Z}_2 -even [12]. From the unitarity considerations it has been shown [12] that the masses of these extra physical scalars are below 1 TeV.

Like most of the extended scalar sector models here also the FCNC related issues are to be dealt carefully. Even though a dedicated FCNC study of this model is beyond the ambit of this paper, we make a few comments regarding this. In the alignment limit the particle h will have exact SM-like couplings, and will not generate any tree-level FCNC. However, the other neutral scalars will in general have tree-level FCNCs, and their masses and couplings can be constrained from flavor data. For example, the \mathbb{Z}_2 -even scalars other than h can induce FCNC involving the first two generations of quarks, thereby contributing to K^0 - \bar{K}^0 oscillation. Similarly, \mathbb{Z}_2 -odd neutral scalars will be constrained by neutral B -meson oscillation data.

Earlier, the signatures of h^0 were studied [19,20], but the residual \mathbb{Z}_2 symmetry was not identified and thus the generic behavior of other similar scalars was not realized properly. For example, a light enough h^0 state can be probed in the $t \rightarrow ch^0$ channel whereas a heavier h^0 can manifest itself in the channel $h^0 \rightarrow (t\bar{c} + c\bar{t})$. It is also worth mentioning that in the exact \mathbb{Z}_2 limit, we do not have $\bar{t}th^0$ coupling and in the S_3 alignment limit [12], h^0VV ($V = W, Z$) coupling also vanishes. But it might be possible to produce h^0 via the coupling with the SM-like Higgs (h^0h^0h). Of course, for testing any such outcome, it will have to be remembered that the \mathbb{Z}_2 symmetry is violated by soft terms in the Lagrangian, so the processes forbidden by the \mathbb{Z}_2 symmetry will actually occur, although with a very small rate proportional to powers of Δ/v . The decay of SM-like Higgs, $h \rightarrow \gamma\gamma$ can also be useful in the sense that a precise measurement of this diphoton signal strength can put constraints on the charged scalar masses as they are not decoupled [12,21] even when their masses lie in the TeV range.

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